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## ON BOUNDED SOLUTIONS OF A PROBLEM OF R. SCHILLING

JANUSZ MORAWIEC

**Abstract.** It is proved that if

$$0 < q \leq (1 - \sqrt[3]{2} + \sqrt[3]{4})/3,$$

then the zero function is the only solution  $f: \mathbb{R} \rightarrow \mathbb{R}$  of (1) satisfying (2) and bounded in a neighbourhood of at least one point of the set (3).

The paper concerns bounded solutions  $f: \mathbb{R} \rightarrow \mathbb{R}$  of the functional equation

$$(1) \quad f(qx) = \frac{1}{4q} [f(x-1) + f(x+1) + 2f(x)]$$

such that

$$(2) \quad f(x) = 0 \quad \text{for } |x| > Q$$

where  $q$  is a fixed number from the open interval  $(0, 1)$  and

$$Q = \frac{q}{1-q}.$$

In what follows any solution  $f: \mathbb{R} \rightarrow \mathbb{R}$  of (1) satisfying (2) will be called a *solution of Schilling's problem*. In the present paper we are interested in bounded solutions of Schilling's problem. The first theorem in this direction was obtained by K. Baron in [1]. This theorem reads as follows:

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If  $q \in (0, \sqrt{2} - 1]$ , then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of the origin.

This paper generalizes the above theorem in two directions. Namely, the interval  $(0, \sqrt{2} - 1]$  is replaced by the larger one  $(0, \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}]$  and instead of the boundedness in a neighbourhood of the origin we have boundedness in a neighbourhood of at least one point of the set

$$(3) \quad \left\{ \varepsilon \sum_{i=1}^n q^i : n \in \mathbf{N} \cup \{0, +\infty\}, \varepsilon \in \{-1, 1\} \right\}.$$

(To simplify formulas we adopt the convention  $\sum_{i=1}^0 a_i = 0$  for all real sequences  $(a_i : i \in \mathbf{N})$ .) In other words, we shall prove the following.

THEOREM. If

$$(4) \quad 0 < q \leq \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3},$$

then the zero function is the only solution of Schilling's problem which is bounded in a neighbourhood of at least one point of the set (3).

The proof of this theorem is based on two lemmas. However, we start with the following simple remarks.

REMARK 1. If  $f$  is a solution of Schilling's problem then so is the function  $g: \mathbb{R} \rightarrow \mathbb{R}$  defined by the formula  $g(x) = f(-x)$ .

REMARK 2. Assume that  $f$  is a solution of Schilling's problem.

(i) If  $q \neq \frac{1}{4}$ , then  $f(Q) = 0$ . If  $q = \frac{1}{4}$ , then  $f(Q) = 0$  iff  $f(qQ) = 0$ .

(ii) If  $q < \frac{1}{2}$ , then  $f(0) = 0$ .

PROOF. It is enough to put in (1):  $x = Q/q$ ,  $x = Q$  and  $x = 0$ , respectively, and to use condition (2).  $\square$

LEMMA 1. Assume  $q \in (0, \frac{1}{2})$ . If a solution of Schilling's problem vanishes either on the interval  $(-q, 0)$  or on the interval  $(0, q)$ , then it vanishes everywhere.

PROOF. Let  $f$  be a solution of Schilling's problem vanishing on the interval  $[0, q)$ . We shall prove that  $f$  vanishes on the interval  $[0, Q)$ . Define a sequence of sets  $(A_n : n \in \mathbf{N})$  by the formula

$$A_n = [0, \sum_{i=1}^n q^i).$$

Fix a positive integer  $n$  and suppose that  $f$  vanishes on the set  $A_n$ . We shall show that  $f$  vanishes also on the set  $A_{n+1}$ . To this end fix an  $x_0 \in A_{n+1} \setminus A_n$ . Putting  $x = x_0/q$  into (1) and taking into account that  $x - 1 \in A_n$ , whereas  $x + 1 > x > 1 > Q$  we get

$$(5) \quad f(x_0) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] = 0.$$

Consequently,  $f$  vanishes on the set  $\bigcup_{n=1}^{\infty} A_n$  which equals to  $[0, Q)$ . This and Remark 2 (i) show that  $f$  vanishes on  $[0, +\infty)$ . Hence and from (1) we infer that  $f$  vanishes everywhere.

The case of the interval  $(-q, 0)$  reduces to the previous one by using Remark 1.  $\square$

LEMMA 2. Assume  $q \in (0, \frac{1}{2})$ . If  $f$  is a solution of Schilling's problem, then

$$(6) \quad f\left(q^{m+n}x + \varepsilon \sum_{i=1}^n q^i\right) = \left(\frac{1}{2}\right)^n \left(\frac{1}{2q}\right)^{m+n} f(x)$$

for all  $x \in (Q-1, 1-Q)$ , for all  $\varepsilon \in \{-1, 1\}$ , and for all non-negative integers  $m$  and  $n$ .

PROOF. Fix an  $x_0 \in (Q-1, 1-Q)$ . First we shall show that

$$(7) \quad f(q^m x_0) = \left(\frac{1}{2q}\right)^m f(x_0)$$

for all non-negative integers  $m$ . Of course (7) holds for  $m = 0$ . Suppose that (7) holds for an  $m$ . Putting  $x = q^m x_0$  into (1) and using (2) and (7) we have

$$\begin{aligned} f(q^{m+1} x_0) &= \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] = \frac{1}{2q}f(x) \\ &= \left(\frac{1}{2q}\right)^{m+1} f(x_0). \end{aligned}$$

This proves that (7) holds for all non-negative integers  $m$ .

Fix now a non-negative integer  $n$  and suppose that (6) is satisfied for all  $x \in (Q-1, 1-Q)$ , for all  $\varepsilon \in \{-1, 1\}$ , and for all non-negative integers  $m$ . Putting  $x = q^{m+n} x_0 + \varepsilon \sum_{i=1}^n q^i + \varepsilon$  into (1) and applying (2) and (6) with  $x = x_0$  we obtain

$$\begin{aligned} f\left(q^{m+n+1} x_0 + \varepsilon \sum_{i=1}^{n+1} q^i\right) &= f(qx) = \frac{1}{4q}[f(x-1) + f(x+1) + 2f(x)] \\ &= \frac{1}{4q}f(x-\varepsilon) = \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2q}\right)^{m+n+1} f(x_0). \end{aligned}$$

The proof is completed.  $\square$

Now we pass to the proof of the main theorem.

**PROOF OF THE THEOREM.** It follows from (4) that  $q < 1/2$ .

Fix  $n \in \mathbb{N} \cup \{0, +\infty\}$  and  $\varepsilon \in \{-1, 1\}$  such that a solution  $f$  of Schilling's problem is bounded in a neighbourhood of  $\varepsilon \sum_{i=1}^n q^i$ . We may (and we do) assume that  $n$  is finite.

If  $|x| < 1 - Q$  is fixed, then the left-hand-side of (6) is bounded with respect to  $m$  whereas  $\lim_{m \rightarrow \infty} (1/2q)^{m+n} = +\infty$ . This shows that

$$(8) \quad f(x) = 0 \quad \text{for } |x| < 1 - Q.$$

Consider two cases:

$$(i) \quad q \leq \frac{3 - \sqrt{5}}{2}$$

and

$$(ii) \quad \frac{3 - \sqrt{5}}{2} < q \leq \frac{1}{3} - \frac{\sqrt[3]{2}}{3} + \frac{\sqrt[3]{4}}{3}.$$

In the case (i) we have  $q \leq 1 - Q$  which jointly with (8) and Lemma 1 gives  $f = 0$ .

So we assume now that (ii) holds. First we notice that putting  $x = 1 - Q$  into (1) and applying (8), Remarks 1 and 2(i) and (2) we get

$$0 = f(q(1 - Q)) = \frac{1}{4q}[f(-Q) + f(2 - Q) + 2f(1 - Q)] = \frac{1}{2q}f(1 - Q).$$

Hence, from (8) and Remark 1 we obtain

$$(9) \quad f(x) = 0 \quad \text{for } |x| \leq 1 - Q.$$

Fix an  $x_0 \in [qQ, q(2 - Q)]$ . Putting  $x = x_0/q$  into (1) and using (9), (2) and Remark 2 we have (5). Similarly (cf. Remark 1),  $f(x) = 0$  for  $x \in [-q(2 - Q), -qQ]$ . Consequently,

$$(10) \quad f(x) = 0 \quad \text{whenever } qQ \leq |x| \leq q(2 - Q).$$

Now we fix an  $x_0 \in [q - q^2(2 - Q), q^2(2 - Q)]$ . Putting  $x = x_0/q$  into (1), taking into account the inequality  $qQ < 1 - q(2 - Q)$  and applying (10) and (2) we obtain (5) once again. Similarly  $f(x) = 0$  for  $x \in [-q^2(2 - Q), -q + q^2(2 - Q)]$  and so

$$(11) \quad f(x) = 0 \quad \text{whenever } q - q^2(2 - Q) \leq |x| \leq q^2(2 - Q).$$

As the function  $3t^3 - 3t^2 + 3t - 1$  increases and vanishes at  $(1 - \sqrt[3]{2} + \sqrt[3]{4})/3$ , we have

$$(12) \quad q - q^2(2 - Q) \leq 1 - Q.$$

Relations (9), (12) and (11) give

$$(13) \quad f(x) = 0 \quad \text{for} \quad |x| \leq q^2(2 - Q).$$

Now let us fix an  $x_0 \in [1 - q(2 - Q), 1 - qQ]$ . Putting  $x = x_0 - 1$  into (1) and using (13), (2) and (10) we have

$$(14) \quad 0 = f(qx) = \frac{1}{4q}[f(x - 1) + f(x + 1) + 2f(x)] = \frac{1}{4q}f(x_0).$$

So we obtain

$$(15) \quad f(x) = 0 \quad \text{whenever} \quad 1 - q(2 - Q) \leq x \leq 1 - qQ.$$

Since (cf. (12))  $q + q^2Q < 1 - qQ$  and  $1 - q(2 - Q) < q(2 - Q)$ , (15) proves that

$$(16) \quad f(x) = 0 \quad \text{whenever} \quad q(2 - Q) \leq x \leq q + q^2Q.$$

Finally assume that  $1 - Q \leq x_0 \leq qQ$ . Putting  $x = x_0 + 1$  into (1) and using (16) and (2) we see that (14) holds. Hence

$$f(x) = 0 \quad \text{whenever} \quad 1 - Q \leq x \leq qQ,$$

which jointly with (9) and (10) gives

$$f(x) = 0 \quad \text{whenever} \quad 0 \leq x \leq q(2 - Q).$$

In particular, since  $q < q(2 - Q)$ ,  $f$  vanishes on the interval  $(0, q)$ . This jointly with Lemma 1 completes the proof.  $\square$

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#### REFERENCES

- [1] K. Baron, *On a problem of R. Schilling*, Berichte der Mathematisch-statistischen Sektion in der Forschungsgesellschaft Joanneum-Graz, Bericht 286 (1988).

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